Chapter 2

Derivatives

Chapter2: DERIVATIVES

(I) Derivative of a function at a point

Definition: The derivative of a function $y = f(x)$ at a point a, denoted by $f'(a)$, is

$$\boxed{f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad \text{or} \quad f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}}$$

Provided that the limit exists.

Remark : The value of this limit $f'(a)$, if it exists, represents:

1) the slope of the line tangent to the curve $y = f(x)$ at the point $x = a$. Hence, the equation of tangent line for this curve at a point *a is given by*

$$\frac{y-f(a)}{x-a} = m = f'(a)$$

2) the instantaneous rate of change, with respect to x, of the function f at a. Therefore, a positive $f'(a)$ means that the function f is *increasing* at a, while a negative $f'(a)$ means that f is *decreasing* at a. If $f'(a) = 0$, then f is neither increasing nor decreasing at a.

Example: Let $f(t) = t^5 + 6t$, find $f'(a)$ using the definition of differentiation. .

Solution: Since Let $f(t) = t^5 + 6t$*, then we have*

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{[(a+h)^5+6(a+h)]-[a^5+6a]}{h}$$

$$= \lim_{h\to 0}\frac{[a^5+5a^4h+10a^3h^2+10a^2h^3+5ah^4+h^5+6a+6h]-[a^5+6a]}{h}$$

$$= \lim_{h\to 0}\frac{5a^4h+10a^3h^2+10a^2h^3+5ah^4+h^5+6h}{h}$$

$$= \lim_{h \to 0} \frac{h(5a^4 + 10a^3h + 10a^2h^2 + 5ah^3 + h^4 + 6)}{h}$$

$$= \lim_{h \to 0} (5a^4 + 10a^3h + 10a^2h^2 + 5ah^3 + h^4 + 6) = 5a^4 + 6$$

Example: Let $f(x) = \sqrt{4x^2 + 5}$, find $f'(a)$ using the definition of differentiation. Write an equation of the line tangent to $y = f(x)$ when $a = 1$.

Solution:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{4(a+h)^2 + 5} - \sqrt{4a^2 + 5}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{4(a+h)^2 + 5} - \sqrt{4a^2 + 5}}{h} \cdot \frac{\sqrt{4(a+h)^2 + 5} + \sqrt{4a^2 + 5}}{\sqrt{4(a+h)^2 + 5} + \sqrt{4a^2 + 5}}$$

$$= \lim_{h \to 0} \frac{4(a+h)^2 + 5 - (4a^2 + 5)}{h\left(\sqrt{4(a+h)^2 + 5} + \sqrt{4a^2 + 5}\right)} = \lim_{h \to 0} \frac{4a^2 + 8ah + 4h^2 + 5 - 4a^2 - 5}{h\left(\sqrt{4(a+h)^2 + 5} + \sqrt{4a^2 + 5}\right)}$$

$$= \lim_{h \to 0} \frac{8ah + 4h^2}{h\left(\sqrt{4(a+h)^2 + 5} + \sqrt{4a^2 + 5}\right)} = \lim_{h \to 0} \frac{8a + 4h}{\sqrt{4(a+h)^2 + 5} + \sqrt{4a^2 + 5}}$$

$$= \frac{8a + 4(0)}{\sqrt{4(a+0)^2 + 5} + \sqrt{4a^2 + 5}} = \frac{8a}{2\sqrt{4a^2 + 5}} = \frac{4a}{\sqrt{4a^2 + 5}}$$

At $a = 1$, the point on the curve, $(a, f(a))$ is $(1, 3)$, and the slope of the tangent line is $f'(1) = \frac{4}{3}$. Hence, the equation of tangent line becomes

$$y - 3 = \frac{4}{3}(x - 1)$$

Example: Let $f(x) = \frac{1}{\sqrt{x^2 + 3}}$, using the definition find $f'(a)$.

Solution:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\dfrac{1}{\sqrt{(a+h)^2+3}}-\dfrac{1}{\sqrt{a^2+3}}}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{1}{\sqrt{(a+h)^2+3}}-\dfrac{1}{\sqrt{a^2+3}}}{h}\cdot\frac{\sqrt{(a+h)^2+3}\sqrt{a^2+3}}{\sqrt{(a+h)^2+3}\sqrt{a^2+3}}$$

$$=\lim_{h\to 0}\frac{\sqrt{a^2+3}-\sqrt{(a+h)^2+3}}{h(\sqrt{(a+h)^2+3})(\sqrt{a^2+3})}$$

$$=\lim_{h\to 0}\frac{\sqrt{a^2+3}-\sqrt{(a+h)^2+3}}{h(\sqrt{(a+h)^2+3})(\sqrt{a^2+3})}\cdot\frac{\sqrt{a^2+3}+\sqrt{(a+h)^2+3}}{\sqrt{a^2+3}+\sqrt{(a+h)^2+3}}$$

$$=\lim_{h\to 0}\frac{(a^2+3)-(a^2+2ah+h^2+3)}{h(\sqrt{(a+h)^2+3})(\sqrt{a^2+3})(\sqrt{a^2+3}+\sqrt{(a+h)^2+3})}$$

$$=\lim_{h\to 0}\frac{-2ah-h^2}{h(\sqrt{(a+h)^2+3})(\sqrt{a^2+3})(\sqrt{a^2+3}+\sqrt{(a+h)^2+3})}$$

$$=\lim_{h\to 0}\frac{-2a-h}{(\sqrt{(a+h)^2+3})(\sqrt{a^2+3})(\sqrt{a^2+3}+\sqrt{(a+h)^2+3})}$$

$$=\frac{-2a}{(\sqrt{a^2+3})(\sqrt{a^2+3})(2\sqrt{a^2+3})}$$

$$=\frac{-a}{(a^2+3)^{3/2}}.$$

The Derivative as a function

The derivative of a function of x *is another function of* x. Up until this point, derivatives of functions were calculated at some arbitrary, but fixed, point a. Notice from the previous examples that the expressions obtained can be evaluated at different values of a. Indeed, we can replace the number a in a derivative by the variable x in the expression, and represent the derivative as a function of x.

Definition: The derivative of a function f is the function f', defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for all x for which this limit exists. In this case, $f'(x)$ is called the first derivative of $f(x)$

Remark:

1) The domain of f' is the set of all values from the domain of f where the above limit exists. The process of finding the derivative of f is called <u>differentiation of f</u>. Geometrically, the value of $f'(x)$ represents the slope of the line tangent to the curve
$y = f(x)$ at the point $(x, f(x))$.
2) If a is a number in the domain of f where the derivative exists, then f is said to be <u>differentiable</u> at a.
3) A function is said to be <u>differentiable on an open interval</u> (a, b) if it is differentiable at every point in the interval. For closed intervals, the limit definition of differentiability at an endpoint is replaced by the appropriate one-sided limit.

***Notations*:** Suppose $y = f(x)$, then its derivative with respect to x, is commonly denoted by

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) \; D f(x) = D_x f(x).$$

The symbols $\frac{d}{dx}$ and D are called *differential operators*.

Example: Differentiate $f(x) = x^3 - 7x + 4$, using the definition of differentiation.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - 7(x+h) + 4] - [x^3 - 7x + 4]}{h}$$

$$= \lim_{h \to 0} \frac{[(x^3 + 3x^2 h + 3xh^2 + h^3) - 7x - 7h + 4] - [x^3 - 7x + 4]}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3 - 7h}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 7) = 3x^2 - 7$$

Example: Show that $f(x) = |x|$ is not differentiable at $x = 0$:

Solution: Let $x = 0$ and use the limit definition of derivative,

$$\lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = -1, \qquad \text{and} \qquad \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = 1.$$

Since the one-sided limits are not equal, the limit does not exist, so
$f(x) = |x|$ is not differentiable at 0.

Theorem: If f is differentiable at a, then f is continuous at a.

Remark : The converse is not always true. A function can be continuous at a, but not differentiable at a. For instance, see $f(x) = |x|$, at $x = 0$. Hence, if f is differentiable on an interval, then it is continuous on the same interval as wel

(III) Derivatives of some important real valued functions:

(1) Basic Differentiation Formulas

Here, we give the basic differentiation formulas which are the more important differentiation rules and will allow us to differentiate a wider variety of functions. Suppose f and g are differentiable functions, c is any real number, then

1) $\frac{d}{dx}(c) = 0$

2) $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

3) $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

4) $\frac{d}{dx}[c\,f(x)] = c\cdot\frac{d}{dx}f(x)$

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] = f(x)g'(x) + g(x)f'(x)$$

5) $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

6) Chain rule: if y=g(t) and t=g(x), then

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=f\,'(t)g'(x)=f\,'(g(x))g'(x)$$

7)

(2) Derivatives of power functions

The Power Rule: For any real number *n*,

$$\frac{d}{dx}(x^n)=nx^{n-1}$$

Using the chain rule, we have

$$\boxed{\frac{d}{dx}([u(x)]^n)=nu^{n-1}\frac{du}{dx}}$$

Example: Differentiate $y = 2t^3 - t^{\pi} + t^{-2} + 9$

Solution: $y' = 2(t^3)' - (t^{\pi})' + (t^{-2})' + (9)'$

$= 2(3t^2) - \pi t^{\pi-1} + (-2t^{-3}) + 0$

$= 6t^2 - \pi t^{\pi-1} - 2t^{-3}$

Example: Differentiate $s(t)=5\sqrt{t}-\frac{2}{\sqrt{t}}+4\sqrt[3]{t}$

Solution: This would be easier to do if we first rewrite $s(t)$ in terms of powers of x.

$s(t)=5t^{1/2}-2t^{-1/2}+4t^{1/3}$, then

$$s'(t)=5\left(\frac{1}{2}t^{(\frac{1}{2}-1)}\right)-2\left(\frac{-1}{2}t^{(\frac{-1}{2}-1)}\right)+4\left(\frac{1}{3}t^{(\frac{1}{3}-1)}\right)$$

$$=\frac{5}{2}t^{\frac{-1}{2}}+t^{\frac{-3}{2}}+\frac{4}{3}t^{\frac{-2}{3}}$$

Example: Differentiate $f(x)=\frac{x^2+3x+2}{x^2-3x+2}$

Solution: Using the properties of differentiation, we have

$$f'(x) = \frac{(x^2-3x+2)(x^2+3x+2)' - (x^2+3x+2)(x^2-3x+2)'}{(x^2-3x+2)^2}$$

$$= \frac{(x^2-3x+2)(2x+3) - (x^2+3x+2)(2x-3)}{(x^2-3x+2)^2}$$

Example: Differentiate $g(x) = \dfrac{2x^3+3x^2-x+5}{x^2}$

Solution: The easiest way to do this is to rewrite $g(x)$ as

$g(x) = 2x+3-\dfrac{1}{x}+\dfrac{5}{x^2}$, then

$$\begin{aligned} g'(x) &= 2(x)' + (3)' - (x^{-1})' + 5(x^{-2})' \\ &= 2+0-(-x^{-2})+5(-2x^{-3}) \\ &= 2+x^{-2}-10x^{-3} \\ &= 2+\frac{1}{x^2}-\frac{10}{x^3} \end{aligned}$$

Example: Differentiate $y = \sqrt{x}\,(x^2-5x+2)$

Solution: Simplify first: $y = x^{5/2} - 5x^{3/2} + 2x^{1/2}$.

$$\begin{aligned} y' &= \frac{5}{2}x^{3/2} - 5\left(\frac{3}{2}x^{1/2}\right) + 2\left(\frac{1}{2}x^{-1/2}\right) \\ &= \frac{5}{2}x^{3/2} - \frac{15}{2}x^{1/2} + x^{-1/2} \end{aligned}$$

The longer way to do this is by using the product rule:

$$y' = \sqrt{x}\,(x^2-5x+2)' + (x^2-5x+2)(\sqrt{x})' = \sqrt{x}\,(2x-5) + (x^2-5x+2)(\frac{1}{2}x^{-1/2})$$

$$= x^{1/2}(2x-5)+(x^2-5x+2)(\frac{1}{2}x^{-1/2}) = 2x^{3/2}-5x^{1/2}+\frac{1}{2}x^{3/2}-\frac{5}{2}x^{1/2}+x^{-1/2}$$

$$= \frac{5}{2}x^{3/2}-\frac{15}{2}x^{1/2}+x^{-1/2}$$

Example: Suppose the curves $y_1 = x^2 + ax + b$ and $y_2 = cx - x^2$ have a common tangent line at the point (1, 0). Find the constants a, b, and c.

Solution: Both curves have a common point at (1, 0). Therefore, when $x = 1$, both y-values are 0. Hence, $0 = 1 + a + b$ and $0 = c - 1$. Hence $c = 1$ and $a + b = -1$.

Sharing a tangent line at (1, 0) means that both curves have the same instantaneous rate of change when $x = 1$, i.e., $y_1'(1) = y_2'(1)$.

$$y_1' = 2x + a \quad \Rightarrow \quad y_1'(1) = a + 2$$
$$y_2' = c - 2x \quad \Rightarrow \quad y_2'(1) = c - 2$$

Substitute in $c = 1$ and equate $y_1'(1) = y_2'(1)$:

$$y_1'(1) = a + 2 = y_2'(1) = c - 2 = -1$$

Therefore, $a = -3$, $b = 2$, and $c = 1$.

(3) Derivatives of Trigonometric functions:

Let $y = \sin x$. Using the definition of differentiation

$$y' = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

Since, $\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2}$, we obtain

$$y' = \lim_{h\to 0}\frac{2\sin(h/2)\cos(x+h/2)}{h}$$

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\cos(x+\frac{h}{2})\cdot\frac{\sin(h/2)}{h/2} = \cos x$$

If $y = \sin u \;\; u = g(x)$, we have

$$\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$$

Similarly, we

- $\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$
- $\frac{d}{dx}\tan u = \sec^2 u \cdot \frac{du}{dx}$
- $\frac{d}{dx}\cot u = -\cos ec^2 u \cdot \frac{du}{dx}$
- $\frac{d}{dx}\sec u = \sec u \cdot \tan u \cdot \frac{du}{dx}$
- $\frac{d}{dx}\cos ec\, u = -\cos ec u \cdot \cot u \cdot \frac{du}{dx}$

Example: Evaluate y' for each of the following functions:

1) $y = \sin(2-3x)$ 2) $y = \sin^3 x$

3) $y = \tan^4(2x+1)$ 4) $y = x^3 \sec^2 3x$

5) $y = x\sec^4(3+7x)$ 6) $y = (2\tan x + \sin\sqrt{x})^{-5}$

Solution:

1) $y' = -3\cos(2-3x)$

2) $y' = 3\sin^2 x \cos x$

3) $y' = 4\tan^3(2x+1)\cdot 2\sec^2(2x+1)$

$$= 8\tan^3(2x+1)\sec^2(2x+1)$$

4) $y' = x^3 \cdot 2\sec 3x \cdot \frac{d}{dx}\sec(3x) + [\sec(3x)]^2 \cdot 3x^2$

$$= 2x^3 \sec 3x[3\sec 3x \tan 3x] + 3x^2[\sec(3x)]^2$$

$$= 3x^2 \sec^2 3x(2x\tan 3x + 1)$$

5) $y' = \sec^4(3+7x) + 28x\sec^3(3+7x)\cdot\sec(3+7x)$

$$\cdot\tan(3+7x)$$

$$=\sec^4(3+7x)+28x\sec^4(3+7x)\tan(3+7x)$$

6) $y'=-5(2\tan x+\sin\sqrt{x})^{-6}\cdot\left(2\sec^2 x+\dfrac{\cos\sqrt{x}}{2\sqrt{x}}\right)$

(4) Derivatives of Exponential and Logarithm Functions

First, we deal with the derivatives of logarithm function. Let

$$Let\ \ y=\ln x,\ then\ y'\ is\ given\ by$$

$$y'=\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}=\lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right)$$

$$=\lim_{\frac{h}{x}\to 0}\ln\left[\left(1+\frac{h}{x}\right)^{x/h}\right]^{1/x}=\frac{1}{x}\lim_{\frac{h}{x}\to 0}\ln\left(1+\frac{h}{x}\right)^{x/h}$$

$$y'=\frac{1}{x}\ln\lim_{\frac{h}{x}\to 0}\left(1+\frac{h}{x}\right)^{x/h}=\frac{1}{x}\ln e=\frac{1}{x}$$

If $y=\ln u$ and $u=g(x)$, we get by using the chain rule:

$$\boxed{\frac{d}{dx}\ln u=\frac{1}{u}\cdot\frac{du}{dx}}$$

To determine the derivative rule for $y=\log_a x$, we use the relation

$$\log_a x=\frac{\ln x}{\ln a}$$

Therefore, we obtain $\qquad \dfrac{d}{dx}\log_a x=\dfrac{d}{dx}\dfrac{\ln x}{\ln a}=\dfrac{1}{\ln a}\cdot\dfrac{1}{x}$

$$\boxed{\frac{d}{dx}\log_a x=\frac{1}{\ln a}\cdot\frac{1}{x}}$$

Example: Evaluate y' for each of the following functions:

1) $y=\log_a(x^2+4)$ 2) $y=\ln\sqrt{\dfrac{1+\sin x}{1-\sin x}}$ 3) $y=\log_{10}(x^3+1)^{-7}$

4) $y=\ln\sqrt[5]{4+\cot 3x}$ 5) $y=(2+\sec x)^{\tan x}$

Solution:

1) $$y'=\frac{1}{(x^2+4)\ln a}\frac{d}{dx}(x^2+4)=\frac{2x}{(x^2+4)\ln a}$$

2) $$y=\ln\sqrt{\frac{1+\sin x}{1-\sin x}}=\frac{1}{2}\ln(1+\sin x)-\frac{1}{2}\ln(1-\sin x)$$

$$\therefore y'=\frac{1}{2}\cdot\frac{\cos x}{(1+\sin x)}-\frac{1}{2}\cdot\frac{-\cos x}{(1-\sin x)}$$

$$=\frac{\cos x}{2}\left(\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\right)$$

$$=\frac{\cos x}{2}\cdot\frac{2}{1-\sin^2 x}=\frac{\cos x}{\cos^2 x}=\sec x$$

3) $$y'=\frac{-7}{\ln 10}\cdot\frac{3x^2}{x^3+1}$$

4) $$y=\ln\sqrt[5]{4+\cot 3x}=\frac{1}{5}\ln(4+\cot 3x)$$

$$\therefore y'=\frac{1}{5}\cdot\frac{-3\operatorname{cosec}^2 3x}{4+\cot 3x}=-\frac{3\operatorname{cosec}^2 3x}{5(4+\cot 3x)}$$

5) $$y=(2+\sec x)^{\tan x}$$

Taking natural logarithm function ln for both sides of this equation and use its properties, we get

$$\ln y=\tan x\cdot\ln(2+\sec x)$$

$$\frac{y'}{y}=\tan x\cdot\frac{\sec x\tan x}{2+\sec x}+\sec^2 x\cdot\ln(2+\sec x)$$

Hence, using $y=(2+\sec x)^{\tan x}$ we conclude that

$$y'=\left[\frac{\sec x\tan^2 x}{2+\sec x}+\sec^2 x\cdot\ln(2+\sec x)\right]\cdot(2+\sec x)^{\tan x}$$

Now, we deal with the derivatives of exponential function:

Let $y = e^x \quad \Rightarrow \quad x = \ln y$

$$\Rightarrow \quad \frac{y'}{y} = 1$$

$$\therefore y' = y = e^x$$

If $y = e^u$ and $u = g(x)$, we get by using the chain rule:

$$\boxed{\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}}$$

Similarly, one can show that

$$\boxed{\frac{d}{dx} a^u = a^u \cdot \ln a}$$

Example: Evaluate y' for each of the following functions:

1) $y = e^{x^2}$ 2) $y = e^{\cos\sqrt{x}}$ 3) $y = \operatorname{cosec} 2e^{3x}$

Solution:

1) $\because y = e^{x^2} \Rightarrow \therefore y' = 2xe^{x^2}$

2) $\because y = e^{\cos\sqrt{x}} \quad \Rightarrow \therefore y' = e^{\cos\sqrt{x}} \cdot (-\sin\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

3) $\because y = \operatorname{cosec} 2e^{3x} \Rightarrow \quad \therefore y' = -\operatorname{cosec} 2e^{3x} \cot 2e^{3x} \cdot (6e^{3x})$

Example Differentiate each of the following.

(a) $h(z) = \dfrac{2}{\left(4z + e^{-9z}\right)^{10}}$

(b) $f(y) = \sqrt{2y + \left(3y + 4y^2\right)^3}$

(c) $y = \tan\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)$

(d) $g(t) = \sin^3\left(e^{1-t} + 3\sin(6t)\right)$

Solution

(a) In this case let's first rewrite the function in a form that will be a little easier to deal with.

$$h(z) = 2\left(4z + e^{-9z}\right)^{-10}$$

Now, let's start the derivative.

$$h'(z) = -20\left(4z + e^{-9z}\right)^{-11} \frac{d}{dz}\left(4z + e^{-9z}\right)$$

Let's go ahead and finish this example out.

$$h'(z) = -20\left(4z + e^{-9z}\right)^{-11}\left(4 - 9e^{-9z}\right)$$

(b) We'll not put as many words into this example, but we're still going to be careful with this

derivative so make sure you can follow each of the steps here.

$$\begin{aligned}
f'(y) &= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}} \frac{d}{dy}\left(2y + \left(3y + 4y^2\right)^3\right) \\
&= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}}\left(2 + 3\left(3y + 4y^2\right)^2 (3 + 8y)\right) \\
&= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}}\left(2 + (9 + 24y)\left(3y + 4y^2\right)^2\right)
\end{aligned}$$

(c) $y = \tan\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)$

Let's jump right into this one.

$$\begin{aligned}
y' &= \sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)\frac{d}{dx}\left(\left(3x^2\right)^{\frac{1}{3}} + \ln\left(5x^4\right)\right) \\
&= \sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)\left(\frac{1}{3}\left(3x^2\right)^{-\frac{2}{3}}(6x) + \frac{20x^3}{5x^4}\right) \\
&= \left(2x\left(3x^2\right)^{-\frac{2}{3}} + \frac{4}{x}\right)\sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)
\end{aligned}$$

(d) $g(t) = \sin^3\left(e^{1-t} + 3\sin(6t)\right)$

We'll need to be a little careful with this one.

$$g'(t) = 3\sin^2\left(e^{1-t} + 3\sin(6t)\right)\frac{d}{dt}\sin\left(e^{1-t} + 3\sin(6t)\right)$$

$$= 3\sin^2\left(e^{1-t} + 3\sin(6t)\right)\cos\left(e^{1-t} + 3\sin(6t)\right)\frac{d}{dt}\left(e^{1-t} + 3\sin(6t)\right)$$

$$= 3\sin^2\left(e^{1-t} + 3\sin(6t)\right)\cos\left(e^{1-t} + 3\sin(6t)\right)\left(e^{1-t}(-1) + 3\cos(6t)(6)\right)$$

$$= 3\left(-e^{1-t} + 18\cos(6t)\right)\sin^2\left(e^{1-t} + 3\sin(6t)\right)\cos\left(e^{1-t} + 3\sin(6t)\right)$$

(5) Derivatives of inverse trigonometric functions:

First, we find the derivative of $y = \arcsin x := \sin^{-1} x$

$$\therefore x = \sin y \quad \Rightarrow \therefore \frac{dx}{dy} = \cos y$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad |x| \le 1$$

If $y = \sin^{-1} u$ and $u = g(x)$, we get by using the chain rule:

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad |u| \le 1$$

In a similar fashion, develop the formulas for the derivatives of the other 5 inverse trig functions:

- $$\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad |u| \le 1$$
- $$\frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1+u^2}\frac{du}{dx}$$

- $$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}$$
- $$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx} \qquad |u| \geq 1$$
- $$\frac{d}{dx}(\operatorname{cosec}^{-1} u) = \frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx} \qquad |u| \geq 1$$

Remark:

Alternate Notation

There is some alternate notation that is used on occasion to denote the inverse trig functions. This notation is,

$$\sin^{-1} x = \arcsin x \qquad \cos^{-1} x = \arccos x$$
$$\tan^{-1} x = \arctan x \qquad \cot^{-1} x = \operatorname{arccot} x$$
$$\sec^{-1} x = \operatorname{arcsec} x \qquad \csc^{-1} x = \operatorname{arccsc} x$$

Example: Differentiate each of the following functions:

1) $y = \cos^{-1}(\tan x)$ 2) $y = (\tan^{-1} x)^4$ 3) $y = \ln(5 + \tan^{-1} 3x)$

Solution:

1) $y' = -\dfrac{1}{\sqrt{1-\tan^2 x}} \cdot \sec^2 x$

2) $y' = 4(\tan^{-1} x)^3 \cdot \dfrac{1}{1+x^2}$

3) $y' = \dfrac{1}{(5+\tan^{-1} 3x)} \cdot \dfrac{3}{(1+9x^2)}$

Remark: Let $x = \varphi(t),\ y = \psi(t)$ be a parametric curve, then $\dfrac{dy}{dx}$ is given by

$$\boxed{\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}\Rightarrow\therefore\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}}$$

Example: Evaluate $\frac{dy}{dx}$ for the following parametric function at the point $P(9,5)$:

$$x=t^3+1,\quad y=2t+1$$

Solution:

$$\because\ x=t^3+1\ \Rightarrow\ \frac{dx}{dt}=3t^2$$

$$\because\ y=2t+1\ \Rightarrow\ \frac{dy}{dt}=2$$

$$\therefore\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{2}{3t^2}$$

One can show that, at the point $P(9,5)$, we get $t=2$. Consequently,

$$\therefore\left.\frac{dy}{dx}\right|_{t=2}=\left.\frac{2}{3t^2}\right|_{t=2}=\frac{1}{6}$$

Example: Evaluate $\frac{dy}{dx}$ for the following parametric function

$$y=\sin^{-1}t\qquad,\qquad x=\sqrt{1-t^2}$$

Solution: Differentiating each y and x with respect to t, we obtain

$$\because\ y=\sin^{-1}t\ \Rightarrow\ \frac{dy}{dt}=\frac{1}{\sqrt{1-t^2}}$$

$$\because\ x=\sqrt{1-t^2}\ \Rightarrow\ \frac{dx}{dt}=\frac{-t}{\sqrt{1-t^2}}$$

$$\therefore\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{1}{\sqrt{1-t^2}}\cdot\frac{\sqrt{1-t^2}}{-t}=-\frac{1}{t}$$

(5) Derivatives of hyperbolic functions and therirs inverse

The set of functions that we're going to be looking in this section at are the hyperbolic functions. In many physical situations combinations of e^x & e^{-x} arise fairly often. Because of this these combinations are given names. There are six hyperbolic functions and they are defined as follows.

$$\cosh x = \tfrac{1}{2}\left(e^x + e^{-x}\right) \quad \sinh x = \tfrac{1}{2}\left(e^x - e^{-x}\right) \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \operatorname{cosech} x = \frac{1}{\sinh x}, \qquad \tanh x = \frac{1}{\coth x} = \frac{\sinh x}{\cosh x}$$

Because the hyperbolic functions are defined in terms of exponential, then using the derivative rule of exponential function, we have

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

$$\therefore \frac{d}{dx}\sinh x = \cosh x$$

If $y = \sinh u$ and $u = g(x)$, we get by using the chain rule:

$$\boxed{\therefore \frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}}$$

Similarly, one can show that

- $\dfrac{d}{dx}\cosh u = \sinh u \dfrac{du}{dx}$
- $\dfrac{d}{dx}\coth u = -\operatorname{cosech}^2 u \dfrac{du}{dx}$,
- $\dfrac{d}{dx}\tanh u = \operatorname{sech}^2 u \dfrac{du}{dx}$
- $\dfrac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$
- $\dfrac{d}{dx}\operatorname{cosech} u = -\operatorname{cosech} u \coth u \dfrac{du}{dx}$

Now, we are deriving the derivatives rules for the inverse hyperbolic functions:

Let $y = \sinh^{-1} x \Rightarrow x = \sinh y \Rightarrow \frac{dx}{dy} = \cosh y$

$$\therefore \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 x}} = \frac{1}{\sqrt{x^2+1}}$$

If $y = \sinh^{-1} u$ and $u = g(x)$, then we get

$$\boxed{\therefore \frac{dy}{dx} = \frac{1}{\sqrt{u^2+1}} \frac{du}{dx}}$$

Similarly, we obtain the following formulas:

- $\frac{d}{dx} \cosh^{-1} u = \frac{u'}{\sqrt{u^2-1}}$
- $\frac{d}{dx} \tanh^{-1} u = \frac{u'}{1-u^2}$
- $\frac{d}{dx} \coth^{-1} u = -\frac{u'}{1-u^2}$
- $\frac{d}{dx} \operatorname{sec} h^{-1} u = -\frac{u'}{u\sqrt{1-u^2}}$
- $\frac{d}{dx} \operatorname{cos} ech^{-1} u = -\frac{u'}{u\sqrt{1+u^2}}$

Here are a couple of quick derivatives using hyperbolic functions.

Example Differentiate each of the following functions.

(a) $f(x) = 2x^5 \cosh x$

(b) $h(t) = \frac{\sinh t}{t+1}$

Solution

(a)

$$f'(x) = 10x^4 \cosh x + 2x^5 \sinh x$$

(b)

$$h'(t) = \frac{(t+1)\cosh t - \sinh t}{(t+1)^2}$$

Example: Differentiate each of the following functions:

1) $y = \sinh(\tan^2 x)$ 2) $y = \sec h(\ln \sec x)$

Solution:

1) $y' = \cosh(\tan^2 x) \cdot 2\tan x \cdot \sec^2 x$

2) $y' = -\sec h(\ln \sec x) \cdot \tanh(\ln \sec x) \cdot \dfrac{\sec x \tan x}{\sec x}$

Example: Differentiate each of the following functions:

1) $y = \sinh^{-1}(\tan x)$ 2) $y = (\sec x)^{\tanh^{-1} x^2}$

Solution:

1) $y' = \dfrac{1}{\sqrt{\tan^2 x + 1}} \cdot \sec^2 x = \sec x$

2) taking $\ln$ for both sides, we get

$$\ln y = \tanh^{-1} x^2 \cdot \ln(\sec x)$$

$$\frac{y'}{y} = \tanh^{-1} x^2 \cdot \frac{\sec x \tan x}{\sec x} + \ln(\sec x) \cdot \left(\frac{1}{1-x^4}\right) \cdot 2x$$

$$\Rightarrow \quad y' = y[\tanh^{-1} x^2 \cdot \frac{\sec x \tan x}{\sec x} + \ln(\sec x) \cdot \left(\frac{1}{1-x^4}\right) \cdot 2x]$$

$$= (\sec x)^{\tanh^{-1} x^2} [\tanh^{-1} x^2 \cdot \frac{\sec x \tan x}{\sec x} + \ln(\sec x) \cdot \left(\frac{1}{1-x^4}\right) \cdot 2x]$$

To this point we've done quite a few derivatives, but they have all been derivatives of functions
of the form y=f(x). Unfortunately not all the functions that we're going to look at will fall into this form.

Example: Differentiate each of the following functions:

(a) $y = e^t$, $t = \ln \sinh^{-1}(\tan x)$ (b) $x \tan^{-1} y = y \tan^{-1} x$

$$(a)\, y = e^{t} \ \& \ t = \ln \sinh^{-1}(\tan x)$$

$$\Rightarrow \quad y = e^{\ln \sinh^{-1}(\tan x)} = \sinh^{-1}(\tan x)$$

$$= \frac{\sec^2 x}{\sqrt{1+\tan^2 x}}$$

$$(b) \quad \because \ x \tan^{-1} y = y \tan^{-1} x$$

$$\Rightarrow \frac{d}{dx}\left[x \tan^{-1} y\right] = \frac{d}{dx}\left[y \tan^{-1} x\right]$$

$$\left[\tan^{-1} y + x(\tan^{-1} y)'\right] = \left[y' + (\tan^{-1} x)'\right]$$

$$\left[\tan^{-1} y + x\left(\frac{y'}{1+y^2}\right)\right] = \left[y' + \left(\frac{1}{1+x^2}\right)\right]$$

$$y'\left(1 - \frac{x}{1+y^2}\right) = \left(\tan^{-1} y - \frac{1}{1+x^2}\right)$$

$$y' = \left(\tan^{-1} y - \frac{1}{1+x^2}\right) \Big/ \left(1 - \frac{x}{1+y^2}\right)$$

Example Find y' for each of the following.

(a) $x^3 y^5 + 3x = 8y^3 + 1$

(b) $x^2 \tan(y) + y^{10} \sec(x) = 2x$

(c) $e^{2x+3y} = x^2 - \ln(xy^3)$

Solution

(a) Here is the differentiation of each side for this function.

$$3x^2 y^5 + 3 = 24y^2 y' - 5x^3 y^4 y'$$

$$3x^2 y^5 + 3 = \left(24y^2 - 5x^3 y^4\right) y'$$

$$y' = \frac{3x^2 y^5 + 3}{24y^2 - 5x^3 y^4}$$

(b) $x^2\tan(y)+y^{10}\sec(x)=2x$

We've got two product rules to deal with this time. Here is the derivative of this function.

$$2x\tan(y)+x^2\sec^2(y)\,y'+10y^9y'\sec(x)+y^{10}\sec(x)\tan(x)=2$$

Now, solve for the derivative.

$$\left(x^2\sec^2(y)+10y^9\sec(x)\right)y'=2-y^{10}\sec(x)\tan(x)-2x\tan(y)$$

$$y'=\frac{2-y^{10}\sec(x)\tan(x)-2x\tan(y)}{x^2\sec^2(y)+10y^9\sec(x)}$$

(c) Here is the derivative of this equation.

$$e^{2x+3y}(2+3y')=2x-\frac{y^3+3xy^2y'}{xy^3}$$

Now we need to solve for the derivative and this is liable to be somewhat messy. In order to get the y' on one side we'll need to multiply the exponential through the parenthesis and break up the quotient.

$$2e^{2x+3y}+3y'e^{2x+3y}=2x-\frac{y^3}{xy^3}-\frac{3xy^2y'}{xy^3}$$

$$2e^{2x+3y}+3y'e^{2x+3y}=2x-\frac{1}{x}-\frac{3y'}{y}$$

$$\left(3e^{2x+3y}+3y^{-1}\right)y'=2x-x^{-1}-2e^{2x+3y}$$

$$y'=\frac{2x-x^{-1}-2e^{2x+3y}}{3e^{2x+3y}+3y^{-1}}$$

<u>(5) Higher Order Derivatives:</u>

Here we will introduce the idea of higher order derivatives.

<u>Definition:</u> Let $y=f(x)$ be differentiable function, then $y'=\dfrac{dy}{dx}=f'(x)$ is called the first derivative of $f(x)$, then we define the following

$$f''(x)=\frac{d^2f}{dx^2}:=\frac{d}{dx}\left[\frac{df}{dx}\right]\ :\qquad \text{the second derivative of } f(x)$$

$$f'''(x)=\frac{d^3f}{dx^3}:=\frac{d}{dx}\left[f''(x)\right]\ :\qquad \text{the third derivative of } f(x)$$

$$f^{(4)}(x)=f''''(x)=\frac{d^4f}{dx^4}:=\frac{d}{dx}\left[f'''(x)\right]\ :\qquad \text{the fourth derivative of } f(x)$$

$$\vdots$$

$$f^{(n)}(x)=\frac{d^nf}{dx^n}:=\frac{d}{dx}\left[f^{(n-1)}(x)\right]\ :\qquad \text{the } n^{th} \text{ derivative of } f(x).$$

Let's take a look at some examples of higher order derivatives.

Example Find the first four derivatives for each of the following.

(a) $R(t)=3t^2+8t^{\frac{1}{2}}+\mathbf{e}^t$

(b) $y=\cos x$

(c) $f(y)=\sin(3y)+\mathbf{e}^{-2y}+\ln(7y)$

Solution

(a) $R(t)=3t^2+8t^{\frac{1}{2}}+\mathbf{e}^t$

There really isn't a lot to do here other than do the derivatives.

$$R'(t)=6t+4t^{-\frac{1}{2}}+\mathbf{e}^t$$

$$R''(t)=6-2t^{-\frac{3}{2}}+\mathbf{e}^t$$

$$R'''(t)=3t^{-\frac{5}{2}}+\mathbf{e}^t$$

$$R^{(4)}(t)=-\frac{15}{2}t^{-\frac{7}{2}}+\mathbf{e}^t$$

(b) $y = \cos x$

Again, let's just do some derivatives.

$$\begin{aligned} y &= \cos x \\ y' &= -\sin x \\ y'' &= -\cos x \\ y''' &= \sin x \\ y^{(4)} &= \cos x \end{aligned}$$

(c) Similarly, on can show that

$$f'(y) = 3\cos(3y) - 2e^{-2y} + \frac{1}{y} = 3\cos(3y) - 2e^{-2y} + y^{-1}$$

$$f''(y) = -9\sin(3y) + 4e^{-2y} - y^{-2}$$

$$f'''(y) = -27\cos(3y) - 8e^{-2y} + 2y^{-3}$$

$$f^{(4)}(y) = 81\sin(3y) + 16e^{-2y} - 6y^{-4}$$

Example: Find the second derivative for the following.

$$x = a\cos t \quad , \quad y = a\sin t \quad , \quad 0 \le t \le \pi$$

Solution: Using the derivative rule for parametric functions, we get

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{a\cos t}{-a\sin t} = -\cot t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) / \frac{dx}{dt} = \frac{\operatorname{cosec}^2 t}{-a\sin t} = -\frac{1}{a}\operatorname{cosec}^3 t$$

Example: Find the second derivative for the following function $y = e^{kt}$, where k is constant

Solution: Using the derivatives rule for the exponential function, one can show that

$$\because \quad y = e^{kx}$$

$$\Rightarrow \quad y' = ke^{kx} \;,$$

$$\Rightarrow \quad y'' = k^2 e^{kx} \;,$$

$$\Rightarrow \quad y''' = k^3 e^{kx} \;,$$

$$\vdots$$

$$\Rightarrow \quad y^{(n)} = k^n e^{kx}$$

Example: Find the second derivative for the following function $f(x) = \sin ax$

Solution: Using the derivatives rule for the trig function, we get

$$\because f(x) = \sin ax$$

$$\therefore f'(x) = a\cos ax = a\sin(ax + \frac{\pi}{2})$$

$$\Rightarrow f''(x) = a^2\cos(ax + \frac{\pi}{2}) = a^2\sin(ax + 2\frac{\pi}{2})$$

$$\Rightarrow f'''(x) = a^3\cos(ax + \frac{\pi}{2}) = a^3\sin(ax + 3\frac{\pi}{2})$$

$$\vdots$$

$$f^{(n)}(x) = a^n\sin(ax + n\frac{\pi}{2})$$

Example: Find the n^th^ derivative for the following function $y = \ln x$ and hence for $z = \ln x^a$, where a is constant.

Solution: Using the derivatives rule for the trig function, we get

$$\because \quad y = \ln x,$$

$$\Rightarrow y' = +\frac{1}{x}$$

$$\Rightarrow y'' = -\frac{1}{x^2}$$

$$\Rightarrow y''' = +\frac{2}{x^3} = (-1)^2 \frac{2!}{x^3}$$

$$\Rightarrow y^{(4)} = -\frac{2\cdot 3}{x^4} = (-1)^3 \frac{3!}{x^4}$$

$$\vdots$$

$$\Rightarrow y^{(n)} = (-1)^{n-1} \frac{(n-1)!}{x^n}$$

Using the properties of logarithm function, we have

$$\because \quad z = \ln x^a = a \ln x$$

$$\Rightarrow \; z^{(n)} = a[\ln x]^{(n)} = (-1)^{n-1} a \frac{(n-1)!}{x^n}$$

PRACTICES

(1) Find the first derivative for the following functions:

$a)\ y = \dfrac{2+x}{2-x}$ $\qquad$ $b)\ y = (2\sin x - 5x)^{-6}$

$c)\ y = \sqrt{2+\sqrt{x}}$ $\qquad$ $d)\ y = \ln\ln(\sin 5x)$

$e)\ y = \cot(\ln x)$ $\qquad$ $f)\ y = \ln(\sin\sqrt{x})$

$h)\ y = e^{\sin^{-1} x^2}$ $\qquad$ $i)\ y = \cos^{-1}(\cot x)$

$j)\ y = \sin^{-1}\dfrac{2x}{1+x^2}$ $\qquad$ $k)\ y = \tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$

(2) Find the first derivative for the following parametric functions:

1) $x = t^2$, $y = \dfrac{t}{(1-t)^2}$

2) $x = t + \dfrac{1}{t}$, $y = t^n + \dfrac{1}{t^n}$

3) $x = 3\cos\theta - \cos 3\theta$, $y = 3\sin\theta - \sin 3\theta$

(3) Find the second derivative for the following parametric functions:

1) $x = u^3 + u$, $y = u^3 - u$

2) $x = a\cos^3 t$, $y = a\sin^3 t$

3) $x = \sin t - t\cos t$, $y = t\sin t + \cos t$

(4) Find the first derivative for the following implicit functions:

$i)$ $x^{2/3} + y^{2/3} = a^{2/3}$

$ii)$ $1 + xy = e^{xy}$

$iii)$ $\ln(x+y) + x^2 + 3y^3 = 1$

$iv)$ $x\tan^{-1} y = y\tan^{-1} x$

(5)If $y = \sin(\ln x) + \cos(\ln x)$ show that $x^2 y'' + xy' + y = 0$

(6)If $y = 3\cos 2x + 4\sin 2x$ prove that $y'' = -4y$

(7)Find the nth derivative for the following functions:

$a)\ y = \sin 2x$ $b)\ y = \dfrac{1}{2x+5}$ $(c)\ y = \cos ax$

(8)Let $y = x\sin\dfrac{1}{x}$. Find y'' and show that $x^4 y'' + y = 0$